

Calculus I

Chapter 3, Part 1

Dr. Pham Huu Anh Ngoc
Department of Mathematics
International university
Saigon, Vietnam
Email:phangoc@hcmiu.edu.vn

October, 2011

A stylized, handwritten-style logo consisting of the letters 'th.' in a bold, black, cursive font.

Chapter 3. APPLICATIONS OF DIFFERENTIATION



Contents

1. Related Rates
2. Extreme Values
3. The Mean Value Theorem. Shapes of Curves
4. Sketching the Graph of a Function
5. L'Hôpital's Rule
6. Optimization Problems
7. Newton's Method
8. Antiderivatives and Indefinite Integrals

3.1 Related rates



Related rates problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known.

The rate of change is usually with respect to time.

3.1 Related rates

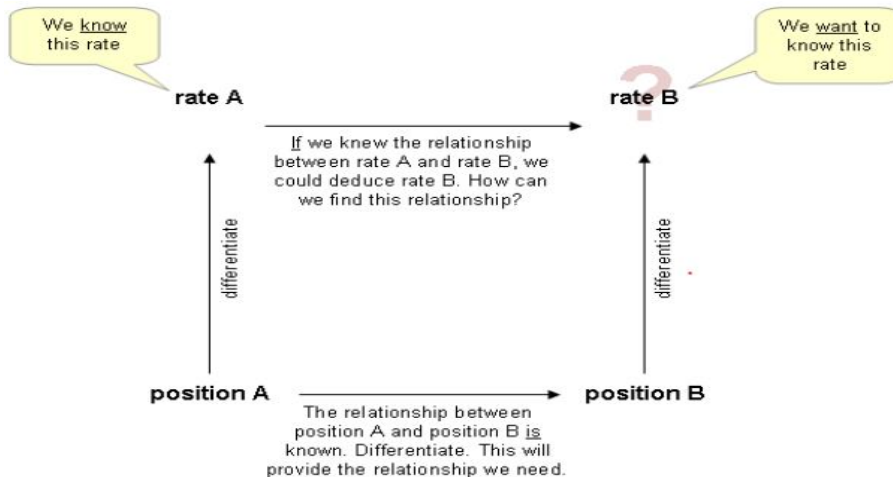


Related rates problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known.

The rate of change is usually with respect to time.

3.1 Related rates

Ch



3.1 Related rates: Example

Ch

Related Rates and Motion A 26-foot ladder is placed against a wall (Fig. 1). If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet away from the wall?



3.1 Related rates: Example

Ch

Related Rates and Motion A 26-foot ladder is placed against a wall (Fig. 1). If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet away from the wall?



3.1 Related rates



Let x be the distance from the bottom of the ladder to the wall and let y be the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time).

We are given that $\frac{dy}{dt} = -2$ feet/ second (y is decreasing at a constant rate of 2 feet per second) and we are asked to find $\frac{dx}{dt}$ when $x = 10$ feet.

In this problem, the relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 26^2.$$

Differentiating each side with respect to t , using the Chain Rule, we have

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

3.1 Related rates



and solving this equation for the desired rate, we obtain

$$\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}.$$

When $x = 10$, the Pythagorean Theorem gives $y = 24$ and so, substituting these values and $\frac{dy}{dt} = -2$, we have

$$\frac{dx}{dt} = -\left(\frac{24}{10}\right)(-2) = 4.8 \text{ feet/second}.$$

Thus the bottom of the ladder is moving away from the wall at a rate of 4.8 feet per second.

3.1 Related rates: Example

Cr

Related Rates and Motion Suppose that two motorboats leave from the same point at the same time. If one travels north at 15 miles per hour and the other travels east at 20 miles per hour, how fast will the distance between them be changing after 2 hours?

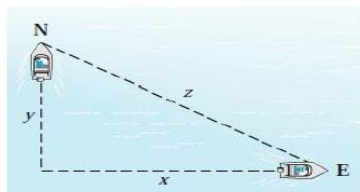


FIGURE 2

3.1 Related rates: Example

Cr

Related Rates and Motion Suppose that two motorboats leave from the same point at the same time. If one travels north at 15 miles per hour and the other travels east at 20 miles per hour, how fast will the distance between them be changing after 2 hours?

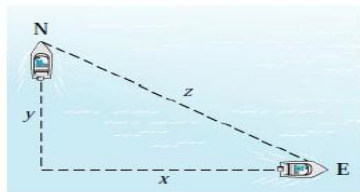


FIGURE 2

Solution: First draw a picture as shown in FIGURE2. All variables x, y, z are changing with t : $x = x(t), y = y(t), z = z(t)$. We have

$$x(t)^2 + y(t)^2 = z(t)^2.$$

3.1 Related rates: Example

Dr

We already know : $\frac{dx}{dt} = 20$; $\frac{dy}{dt} = 15$ and we have to find $\frac{dz}{dt}$ at the end of two hours.

3.1 Related rates: Example



We already know : $\frac{dx}{dt} = 20$; $\frac{dy}{dt} = 15$ and we have to find $\frac{dz}{dt}$ at the end of two hours.

Differentiate the equation $x(t)^2 + y(t)^2 = z(t)^2$ with respect to t , to get

$$2x(t)x'(t) + 2y(t)y'(t) = 2z(t)z'(t).$$

At the end of two hours, we have $x = (20).2 = 40$ and $y = (15).2 = 30$. This implies $z = \sqrt{30^2 + 40^2} = 50$. Finally, we have

$$z' = \frac{2xx' + 2yy'}{2z} = \frac{2.40.20 + 2.30.15}{2.50} = 25 \text{ miles per hour.}$$

3.1 RELATED RATES



Strategy for Solving Related Rate Problems

1. *Draw a picture (if appropriate) and name the variables and constants: Use t for time.*
2. *Write down the numerical information (in terms of the symbols you have chosen).*
3. *Write down what we are asked to find.*
4. *Write an equation that relates the variables.*
5. *Differentiate with respect to t .*
6. *Evaluate, using known values to find the unknown rate.*
7. *Make a concluding statement answering the question asked.*

3.1 RELATED RATES



Example 1.3 When the price of a certain commodity is p dollars per unit, the manufacturer is willing to supply x thousand units, where $x^2 - 2x\sqrt{p} - p^2 = 31$. How fast is the supply changing when the price is \$9 per unit and is increasing at the rate of 20 cents per week?

3.2 EXTREME VALUES

3.2.1 ABSOLUTE EXTREMA



An important problem in many applications is to find the minimum or maximum value of a function.

Calculus provides some very useful tools for solving such problems.

Definition 2.1

Let f be a given function defined on an interval I and $a \in I$. We say that

- $f(a)$ is the **absolute minimum** (or **global minimum**) of f on I if

$$f(a) \leq f(x), \quad \forall x \in I;$$

- $f(a)$ is the **absolute maximum** (or **global maximum**) of f on I if

$$f(a) \geq f(x) \quad \forall x \in I.$$

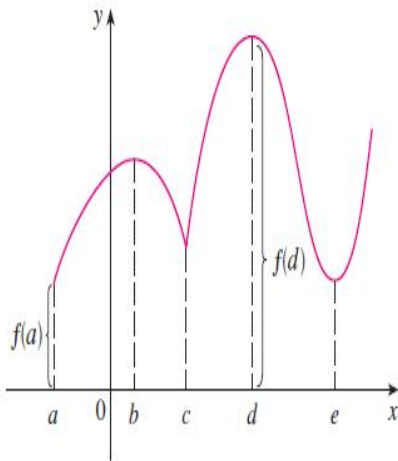
The minimum and maximum values of f on I are called the **extreme values** of f on I .

3.2 EXTREME VALUES

3.2.1 ABSOLUTE EXTREMA

Ch

FIGURE 1
Minimum value $f(a)$,
maximum value $f(d)$



3.2 EXTREME VALUES

3.2.1 ABSOLUTE EXTREMA

Example 2.1 Let $f(x) = \cos x$, $x \in (-\infty, \infty)$. It is clear that

$$-1 \leq f(x) = \cos x \leq 1, \quad \forall x \in (-\infty, \infty),$$

and

$$f((2n+1)\pi) = \cos(2n+1)\pi = -1; \quad f(2n\pi) = \cos(2n\pi) = 1,$$

for any integer n .

Thus the absolute maximum value of f is 1 and the absolute minimum value of f is -1 .

3.2 EXTREME VALUES

3.2.1 ABSOLUTE EXTREMA

cr

Example 2.2 The function $f(x) = x^2$ has the absolute minimum value at $x = 0$ since

$$f(x) = x^2 \geq 0 = f(0) \quad \text{for all } x \in \mathbb{R}.$$

However, there is no highest point on the parabola. So, this function has no maximum value.

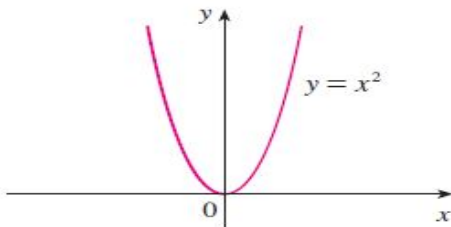


FIGURE 2

Minimum value 0, no maximum

3.2.1 ABSOLUTE EXTREMA

Cr

Consider $f(x) = x^3$, $x \in (-\infty, \infty)$.

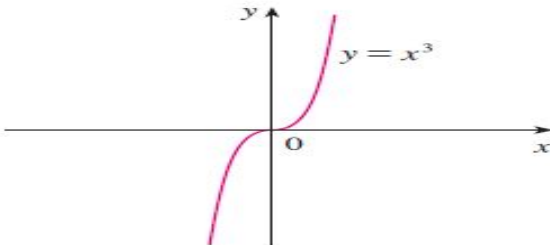


FIGURE 3

No minimum, no maximum

3.2 EXTREME VALUES

3.2.1 ABSOLUTE EXTREMA



Question: When does a function f have an absolute maximum and an absolute minimum ?

3.2 EXTREME VALUES

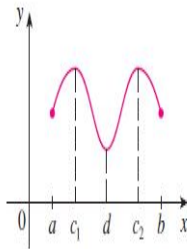
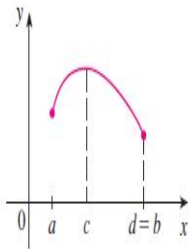
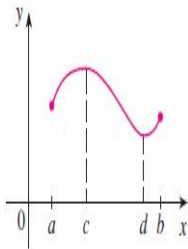
3.2.1 ABSOLUTE EXTREMA

cr

Question: When does a function f have an absolute maximum and an absolute minimum ?

Theorem 2.1

If f is continuous on a **closed interval** $[a, b]$, then f takes on both an absolute minimum and an absolute maximum value on $[a, b]$.



3.2 EXTREME VALUES

3.2.1 ABSOLUTE EXTREMA

cr

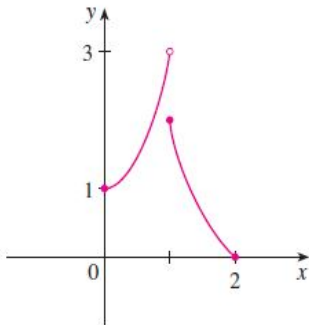


FIGURE 6

This function has minimum value $f(2) = 0$, but no maximum value.

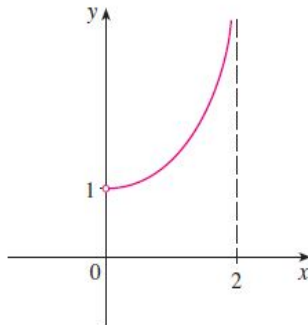


FIGURE 7

This continuous function g has no maximum or minimum.

Definition 2.2

A function f has a **local minimum** (or **relative minimum**) at c if $f(c) \leq f(x)$ for all x near c , that is, if there exists $\delta > 0$ such that

$$f(c) \leq f(x) \quad \text{for all } x \in (c - \delta, c + \delta).$$

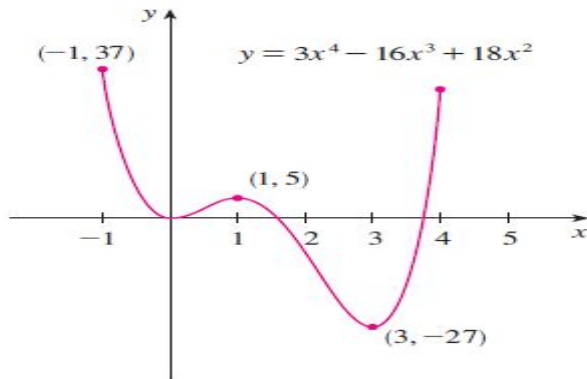
Similarly, f has a **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ for all x near c , that is, if there exists $\delta > 0$ such that

$$f(c) \geq f(x) \quad \text{for all } x \in (c - \delta, c + \delta).$$

Local maxima and minima are called **local extrema**.

3.2.1 LOCAL EXTREMA

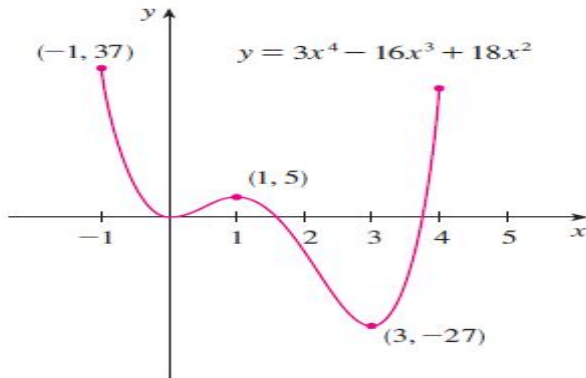
Cr



♠ $f(1) = 5$ is a local maximum;

3.2.1 LOCAL EXTREMA

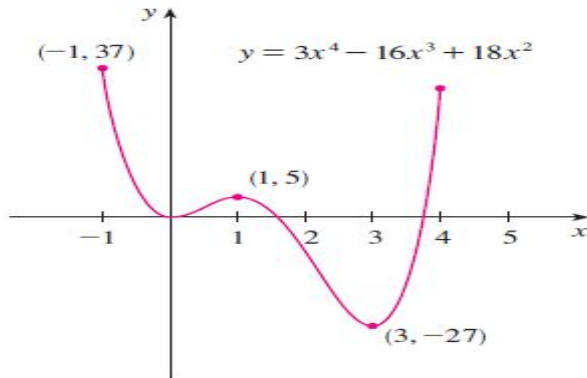
Cr



♠ $f(1) = 5$ is a local maximum; $f(-1) = 37$ is the absolute maximum.

3.2.1 LOCAL EXTREMA

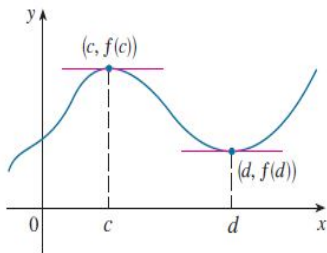
Cr



- ♠ $f(1) = 5$ is a local maximum; $f(-1) = 37$ is the absolute maximum.
- ♠ $f(3) = -27$ is both a local and an absolute minimum.
- ♠ f has neither a local nor an absolute maximum at $x = 4$.

Theorem 2.2 (Fermat's Theorem)

If f has a local maximum or minimum at c and if $f'(c)$ exists, then $f'(c) = 0$.



At the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0. So $f'(c) = 0$ and $f'(d) = 0$.

3.2.2 Local extrema, critical points

Dr

NOTES: The above theorem only offers a necessary condition for a local maximum or minimum point. This is not a sufficient condition.

Example:

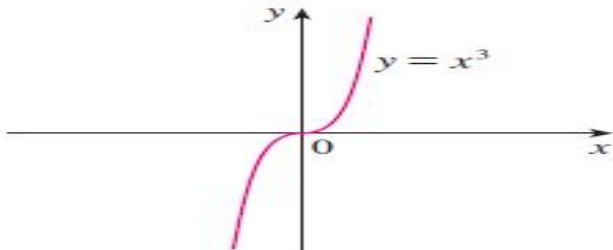


FIGURE 9

If $f(x) = x^3$, then $f'(0) = 0$ but f has no minimum or maximum.

3.2.2 Local extrema, critical points

Cr

Example: The function $f(x) = |x|$ has its minimum value at $x = 0$. However, $f'(0)$ does not exist.

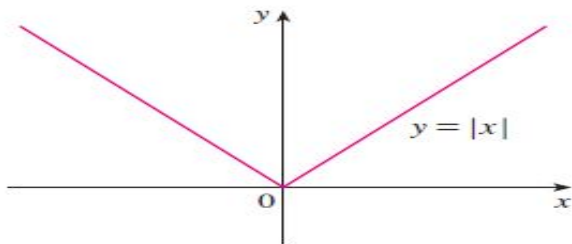


FIGURE 10

If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

Definition 2.3

A **critical number** of a function f is a number in the domain of f such that **either $f'(c) = 0$ or $f'(c)$ does not exist.**

Example 2.4 Find the critical numbers of $f(x) = x^{\frac{3}{5}}(4 - x)$.

Definition 2.3

A **critical number** of a function f is a number in the domain of f such that **either $f'(c) = 0$ or $f'(c)$ does not exist.**

Example 2.4 Find the critical numbers of $f(x) = x^{\frac{3}{5}}(4 - x)$.

SOLUTION The Product Rule gives

$$\begin{aligned} f'(x) &= \frac{3}{5}x^{-2/5}(4 - x) + x^{3/5}(-1) = \frac{3(4 - x)}{5x^{2/5}} - x^{3/5} \\ &= \frac{3(4 - x) - 5x}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}} \end{aligned}$$

Therefore, $f'(x) = 0$ when $x = \frac{3}{2}$ and $f'(x)$ does not exist when $x = 0$.
So the **critical numbers** are 0 and $\frac{3}{2}$.

In terms of critical numbers, Theorem 2.2 can be restated as follows.

If $f(c)$ is a local maximum or minimum, then c must be a critical number of f .

So

The points where a continuous function on a closed interval can take on extreme values are **critical numbers and endpoints**.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b) .
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

3.2 EXTREME VALUES

3.2.2 LOCAL EXTREMA AND CRITICAL POINTS



Example 2.5 Find the maximum and minimum values of $f(x) = \frac{1}{x(1-x)}$ on the interval $[2, 3]$.

3.2 EXTREME VALUES

3.2.2 LOCAL EXTREMA AND CRITICAL POINTS



Example 2.5 Find the maximum and minimum values of $f(x) = \frac{1}{x(1-x)}$ on the interval $[2, 3]$.

Solution: Since f is continuous on $[2, 3]$, f attains the absolute maximum value and the absolute minimum value in this interval.

(Step 1) Find critical numbers. We have

$$f'(x) = \frac{2x - 1}{x^2(1 - x)^2}$$

Therefore, $f'(x) = 0$ when $x = \frac{1}{2}$. Since $\frac{1}{2} \notin [2, 3]$, it is not a critical number.

3.2 EXTREME VALUES

3.2.2 LOCAL EXTREMA AND CRITICAL POINTS



Example 2.5 Find the maximum and minimum values of $f(x) = \frac{1}{x(1-x)}$ on the interval $[2, 3]$.

Solution: Since f is continuous on $[2, 3]$, f attains the absolute maximum value and the absolute minimum value in this interval.

(Step 1) Find critical numbers. We have

$$f'(x) = \frac{2x - 1}{x^2(1 - x)^2}$$

Therefore, $f'(x) = 0$ when $x = \frac{1}{2}$. Since $\frac{1}{2} \notin [2, 3]$, it is not a critical number.

(Step 2) Evaluate $f(x)$ at the endpoints of the closed interval $[2, 3]$:

$$f(2) = -\frac{1}{2}; \quad f(3) = -\frac{1}{6}.$$

(Step 3) The maximum value of f is $f(3) = -\frac{1}{6}$ and the minimum value of f is $f(2) = -\frac{1}{2}$.

3.2 EXTREME VALUES

Ch

Find the maximum and minimum values of
 $f(x) = \cos x + \sin x, \quad x \in [0, 2\pi].$

3.2 EXTREME VALUES



Find the maximum and minimum values of

$$f(x) = \cos x + \sin x, \quad x \in [0, 2\pi].$$

Solution: We find f' :

$$f'(x) = -\sin x + \cos x, \quad x \in [0, 2\pi].$$

Then

$$f'(x) = 0 \Leftrightarrow \tan x = 1, \quad x \in [0, 2\pi].$$

This gives $x = \frac{\pi}{4}; \frac{5\pi}{4}$ (**critical numbers**). Moreover, we have

$$f\left(\frac{\pi}{4}\right) = \sqrt{2}; \quad f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$$

and

3.2 EXTREME VALUES



Find the maximum and minimum values of

$$f(x) = \cos x + \sin x, \quad x \in [0, 2\pi].$$

Solution: We find f' :

$$f'(x) = -\sin x + \cos x, \quad x \in [0, 2\pi].$$

Then

$$f'(x) = 0 \Leftrightarrow \tan x = 1, \quad x \in [0, 2\pi].$$

This gives $x = \frac{\pi}{4}; \frac{5\pi}{4}$ (**critical numbers**). Moreover, we have

$$f\left(\frac{\pi}{4}\right) = \sqrt{2}; \quad f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$$

and

$$f(0) = 1; \quad f(2\pi) = 1 \quad (0, 2\pi \text{ are the endpoints of the interval } [0, 2\pi]).$$

So the maximum of f is $\sqrt{2}$ and the minimum of f is $-\sqrt{2}$.

3.2 Maximum and minimum

Or

Example: Find the maximum and minimum values of the function $f(x) = x^{\frac{2}{3}}$, $x \in [-1, 1]$.

3.2 Maximum and minimum

Cr

Example: Find the maximum and minimum values of the function

$$f(x) = x^{\frac{2}{3}}, \quad x \in [-1, 1].$$

Solution: We find f' :

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3} \frac{1}{x^{\frac{1}{3}}}.$$

Thus, f' is never 0, but is undefined at 0. So $x = 0$ is a critical number. Moreover, we have $f(0) = 0$; $f(-1) = 1$; $f(1) = 1$. So the maximum of f is 1 and the minimum of f is 0.

3.2 Maximum and minimum

Cr

Example: Find the maximum and minimum values of the function $f(x) = x^{\frac{2}{3}}$, $x \in [-1, 1]$.

Solution: We find f' :

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3} \frac{1}{x^{\frac{1}{3}}}.$$

Thus, f' is never 0, but is undefined at 0. So $x = 0$ is a critical number. Moreover, we have $f(0) = 0$; $f(-1) = 1$; $f(1) = 1$. So the maximum of f is 1 and the minimum of f is 0.

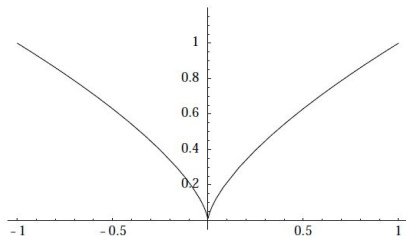


Figure 3.8.3 Graph of $f(x) = x^{\frac{2}{3}}$ on $[-1, 1]$

Theorem 3.1 (Rolle's Theorem)

Suppose that $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) . If

$$f(a) = f(b),$$

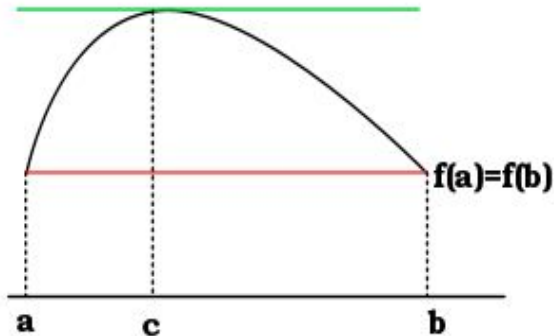
then there exists at least a number c between a and b such that $f'(c) = 0$.

3.3

THE MEAN VALUE THEOREM

tr

3.3.1 ROLLE'S THEOREM AND THE MEAN VALUE THEOREM



The Rolle's theorem states that there exists a point on the graph of f such that the tangent line at that point is parallel to the x-axis.

Example 3.1 The function $f(x) = \frac{1}{3}x^3 - 3x$ is continuous at every point of $[-3, 3]$ and differentiable at every point of $(-3, 3)$. Since

$$f(-3) = 0 = f(3),$$

the Rolle's theorem says that f' must be zero at least once in the open interval between $a = -3$ and $b = 3$.



Example 3.1 The function $f(x) = \frac{1}{3}x^3 - 3x$ is continuous at every point of $[-3, 3]$ and differentiable at every point of $(-3, 3)$. Since

$$f(-3) = 0 = f(3),$$

the Rolle's theorem says that f' must be zero at least once in the open interval between $a = -3$ and $b = 3$.

♠ In fact, $f'(x) = x^2 - 3$ is zero twice in this interval, once at $x = -\sqrt{3}$ and again at $x = \sqrt{3}$.

Example 3.2a Show that the equation $x^3 - 3x + \alpha = 0$ cannot have different roots in $[0, 1]$ for any value of α .

Example 3.2b Consider the quadratic equation:

$$ax^2 + bx + c = 0, \quad (a \neq 0).$$

Suppose that

$$\frac{a}{2012} + \frac{b}{2011} + \frac{c}{2010} = 0.$$

Show that the given quadratic equation has a root in the open interval $(0, 1)$.

Theorem 3.2 (The Mean Value Theorem)

If f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one number c between a and b at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The slope of the secant line AB is

$$m = \frac{f(b) - f(a)}{b - a}.$$

The slope of the tangent line at the point $C = (c, f(c))$ is $f'(c)$. So the Mean Value Theorem says that **there is at least one point on the graph where the the tangent line is parallel to the secant line AB .**



Theorem 3.2 (The Mean Value Theorem)

If f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one number c between a and b at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The slope of the secant line AB is

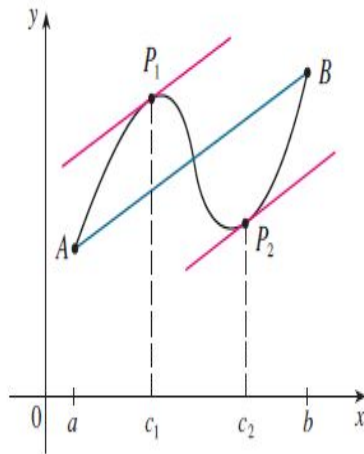
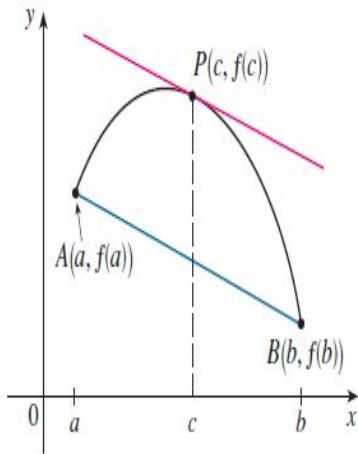
$$m = \frac{f(b) - f(a)}{b - a}.$$

The slope of the tangent line at the point $C = (c, f(c))$ is $f'(c)$. So the Mean Value Theorem says that **there is at least one point on the graph where the the tangent line is parallel to the secant line AB .**

♠ Note that Rolle's Theorem is just the special case of the Mean Value Theorem where $f(a) = f(b)$.

3.3 THE MEAN VALUE THEOREM

Cr



Example 3.3 Show that $\sin x < x$ for all $x > 0$.

Corollary 3.3

If $f'(x) = 0$ for all x in an interval I then f is constant on I .

Recall that a function $f(x)$ is called

(i) **increasing** on an interval I if

$$f(x_1) < f(x_2) \text{ for every } x_1, x_2 \text{ in } I \text{ and } x_1 < x_2;$$

(ii) **decreasing** on an interval I if

$$f(x_1) > f(x_2) \text{ for every } x_1, x_2 \text{ in } I \text{ and } x_1 < x_2.$$

We say that f is **monotonic** on I if it is either increasing or decreasing on I .

Theorem 3.4 (The Sign of the Derivative)

- (a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- (b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example 3.4 Find the intervals on which $f(x) = x^3 - 12x + 5$ is monotonic.



Example 3.4 Find the intervals on which $f(x) = x^3 - 12x + 5$ is monotonic.

 **Solution** We have

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4).$$

So $f'(x) > 0$ if $x < -2$ or $x > 2$ and $f'(x) < 0$ if $-2 < x < 2$. Thus, f is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$ and is decreasing on $(-2, 2)$.

3.3 THE MEAN VALUE THEOREM

3.3.2 INCREASING AND DECREASING FUNCTIONS



Example 3.5 (a) Show that $f(x) = x + e^x$ is increasing on $(-\infty, \infty)$.

(b) Calculate $(f^{-1})'(1)$.

Solution: b) We already know

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}.$$

3.3 THE MEAN VALUE THEOREM

3.3.2 INCREASING AND DECREASING FUNCTIONS



Example 3.5 (a) Show that $f(x) = x + e^x$ is increasing on $(-\infty, \infty)$.

(b) Calculate $(f^{-1})'(1)$.

Solution: b) We already know

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}.$$

Set $f^{-1}(1) = a$. Then $f(a) = 1$, or $a + e^a = 1$.

Example 3.5 (a) Show that $f(x) = x + e^x$ is increasing on $(-\infty, \infty)$.

(b) Calculate $(f^{-1})'(1)$.

Solution: b) We already know

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}.$$

Set $f^{-1}(1) = a$. Then $f(a) = 1$, or $a + e^a = 1$. Since $f(x) = x + e^x$ is increasing on $(-\infty, \infty)$, $x = 0$ is the unique root of the equation

$$1 = x + e^x.$$

Thus $a = 0$ and we have

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2}.$$

Theorem 3.5 (The First Derivative Test)

Suppose that c is a critical point of a continuous function f .

(a) If $f'(x)$ changes from positive to negative at c , then f has a local maximum at c .

(b) If $f'(x)$ changes from negative to positive at c , then f has a local minimum at c .

(c) If $f'(x)$ does not change sign at c , then f has no extremum at c .

Example 3.6 Find the local minimum and maximum values of the function $f(x) = (2x - x^2)e^x$.

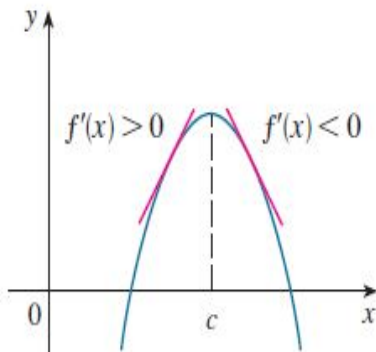
3.3

3.3.2

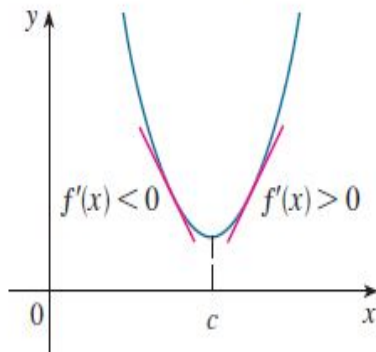
THE MEAN VALUE THEOREM

INCREASING AND DECREASING FUNCTIONS

cr



(a) Local maximum



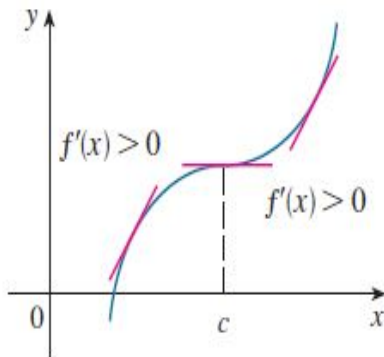
(b) Local minimum

3.3 3.3.2

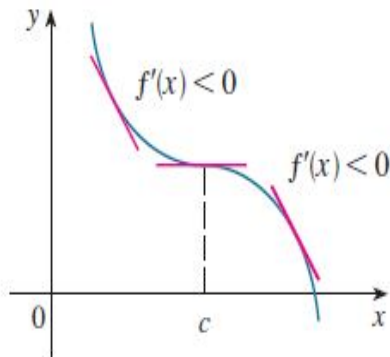
THE MEAN VALUE THEOREM

INCREASING AND DECREASING FUNCTIONS

cr



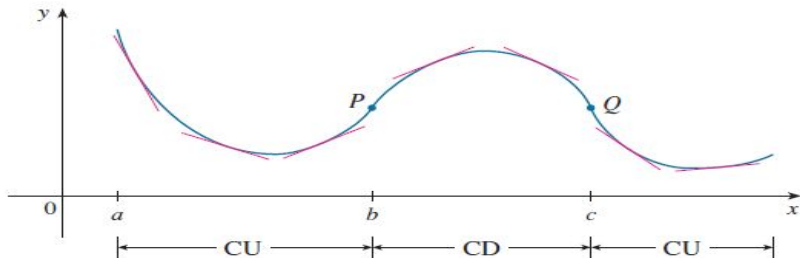
(c) No maximum or minimum



(d) No maximum or minimum

Definition:

A function (or its graph) is called **concave upward** on an interval I if f' is an increasing function on I . It is called **concave downward** on I if f' is decreasing on I .

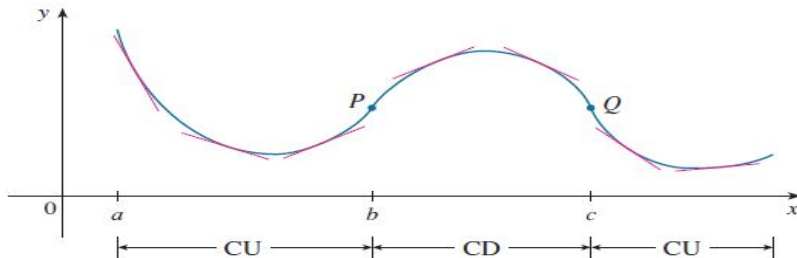


Theorem 3.6 (Test for Concavity)

- (a) If $f''(x) > 0$ for all x in I , then f is concave upward on I .
- (b) If $f''(x) < 0$ for all x in I , then f is concave downward on I .

Definition 3.2

A point where a curve changes its direction of concavity is called an **inflection point**.



The curve in the above figure changes from concave upward to concave downward at P and from concave downward to concave upward at Q , so both P and Q are inflection points.

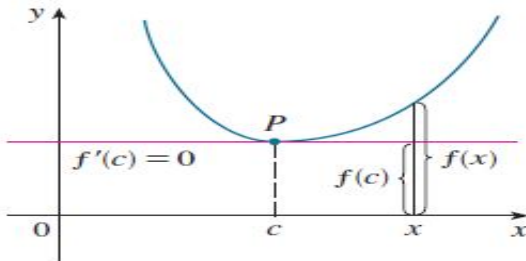
Theorem 3.7 (Test for Inflection Points)

Assume that $f''(x)$ exists for all x in (a, b) and let $c \in (a, b)$. If $f''(c) = 0$ and $f''(x)$ changes sign at $x = c$, then f has an point of inflection at $x = c$.

Example 3.8 Find the points of inflection of $f(x) = x^4 - 4x^3 + 10$ and determine the intervals where $f(x)$ is concave up and down.

The Second Derivative Test Suppose f'' is continuous near c .

- (a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c .
- (b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c .



3.3 THE MEAN VALUE THEOREM

3.3.3 THE SHAPE OF THE GRAPH



Example 3.9 Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Solution: We have

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3); \quad f''(x) = 12x^2 - 24x = 12x(x - 2).$$

To find the **critical numbers** we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. To use the Second Derivative Test we evaluate f'' at these critical numbers

$$f''(0) = 0; \quad f''(3) = 36 > 0.$$

Since $f'(3) = 0$, $f''(3) > 0$, $f(3) = -27$ is a local minimum.

Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0.

3.3 THE MEAN VALUE THEOREM

3.3.3 THE SHAPE OF THE GRAPH

But since $f'(3) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0.

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	+	upward
$(0, 2)$	-	downward
$(2, \infty)$	+	upward

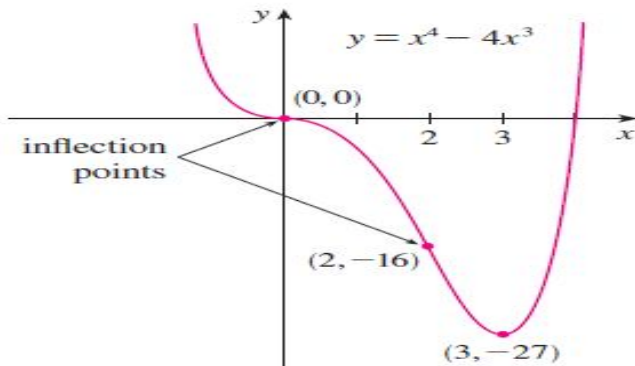
The point $(0, 0)$ is an inflection point since the curve changes from concave upward to concave downward there. Also $(2, -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

3.3 THE MEAN VALUE THEOREM

3.3.3 THE SHAPE OF THE GRAPH

cr

Using the local minimum, the intervals of concavity, and the inflection points, we sketch the curve in the following figure



3.4 SKETCHING THE GRAPH OF A FUNCTION

3.4.1 GRAPH SKETCHING



Example 4.1 Sketch the graph of the function

$$f(x) = x^{2/3}(6 - x)^{1/3}.$$

Example 4.2 Sketch the graph of

$$f(x) = \sqrt[3]{(x^2 - 1)^2}.$$